SyDe 312 - Numerical Methods Unit I Linear Systems

Singular value decomposition supplementary problems

1. Student exploration.

2. Student exploration.

3. $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$ is a 2×3 matrix so we expect the SVD $A = USV^T$ to be have $U(2 \times 2)$, $S(2 \times 3)$, and $V(3 \times 3)$. We can also expect 2 singular values for A, and the S matrix will have a third column of zeros. The matrix $A^TA = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$ has eigenvalues 360, 90 , and 0. The singular values of A are the square roots of the first two eigenvalues: $\sigma_1 = 6\sqrt{10}, \sigma_2 = 3\sqrt{10}$ (conventionally numbered in order of decreasing magnitude). The third zero eigenvalue is irrelevant. Note that the singular values of A must also be square roots of eigenvalues of AA^T, which is a 2×2 matrix, and therefore has only two eigenvalues (the two non-zero eigenvalues of A^TA).

   The first two columns of V are eigenvectors of A^TA corresponding to the non-zero eigenvalues:
   $v_1 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

   The third column of V can be any unit length column vector orthogonal to the first two columns, for instance $v_3 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

   Last calculate the two columns of U. These are obtained from the columns of V corresponding the the non-zero singular values:

$$u_1 = \sigma_1^{-1} A v_1 = \frac{1}{6\sqrt{10}} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$u_2 = \sigma_2^{-1} A v_2 = \frac{1}{3\sqrt{10}} \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix} = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

   So we have:

$$U = \frac{1}{\sqrt{10}} \begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}$$

   The matrix of singular values is:

$$S = \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}$$

The complete SVD is:

$$A = USV^T = \frac{1}{\sqrt{10}}\begin{bmatrix} 3 & -1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix}\begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 2/3 & 1/3 & -2/3 \\ 2/3 & -2/3 & 1/3 \end{bmatrix}$$

Compared to the result of the Matlab SVD function we can see that it is the same except for sign changes, with columns u_1, v_1 and v_3 negatives are our corresponding columns. The SVD is not unique.

4. The matrix $A \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}$ is 3×2 so we expect the SVD $A = USV^T$ to be have $U(3 \times 3)$, $S(3 \times 2)$, and $V(2 \times 2)$. We can also expect 2 singular values for A, and the S matrix will have a third row of zeros.

The product matrix $A^TA = \begin{bmatrix} 9 & -9 \\ -9 & 9 \end{bmatrix}$ has eigenvalues 18 and 0 with corresponding (unit) eigenvectors: $v_1 = \begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix}$ and $v_2 = \begin{bmatrix} -\sqrt{2}/2 \\ -\sqrt{2}/2 \end{bmatrix}$. These eigenvectors form the two columns of V and the square roots of the eigenvalues are the singular values: $\sigma_1 = 3\sqrt{2}, \sigma_2 = 0$.

Last calculate the columns of U. The first column is derived from the non-zero singular value and corresponding column of V:

$$u_1 = \sigma_1^{-1}Av_1 = \frac{1}{3\sqrt{2}}\begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} -\sqrt{2}/2 \\ \sqrt{2}/2 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}$$

The other two columns of U are obtained by extending the first one to form an orthonormal basis of R^3. The easiest way to do this is first to find a vector (x, y, z) that is orthogonal to the first column of U. Taking dot product you get $-x + 2y - 2z = 0$. Solving for a suitable vector put $z = 1$ and $y = 1$ then $x = 2y - 2z = 0$. So the second column of U can be chosen as a unit vector in the direction $(0, 1, 1)$. We'll normalize at the end. To get the third column of U say (x, y, z) it has to be orthogonal to both of the columns already found. Taking dot products you get: $y + z = 0$ and $-x + 2y - 2z = 0$. A solution for this is $(-4, -1, 1)$. Normalizing this vector gives the third column of U.

The U matrix can therefore be chosen to be (4 decimals) $U = \begin{bmatrix} -0.3333 & 0 & 0.9428 \\ 0.6667 & 0.7071 & 0.2357 \\ -0.6667 & 0.7071 & -0.2357 \end{bmatrix}$.

Other vectors could be chosen for the second and third column of U, provided they extend column 1 to be an orthonomal basis of R^3. The choice given above is how Matlab calculates the U matrix (give or take some optional minus signs).

The S matrix would be (5 decimals): $\begin{bmatrix} 4.2426 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

The V matrix is (4 decimals) $V = \begin{bmatrix} -0.7071 & -0.7071 \\ 0.7071 & -0.7071 \end{bmatrix}$.

For the other matrix $A' = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2.1 \end{bmatrix}$ we get the SVD from Matlab:

$$U = \begin{bmatrix} -0.3296 & 0.3023 & 0.8944 \\ 0.6592 & -0.6045 & -0.4472 \\ -0.6759 & -0.7370 & -0 \end{bmatrix}$$

$$S = \begin{bmatrix} 4.2904 & 0 \\ 0 & 0.0521 \\ 0 & 0 \end{bmatrix}$$

$$V = \begin{bmatrix} -0.6992 & 0.7149 \\ 0.7149 & 0.6992 \end{bmatrix}$$

Zeroing the second singular value, without changing U and V, gives:

$$S = \begin{bmatrix} 4.2904 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and then for USV^T we'll have :

$$\begin{bmatrix} 0.9887 & -1.0110 \\ -1.9775 & 2.0220 \\ 2.0275 & -2.0731 \end{bmatrix}$$

This can be compared to the original A and A' matrices as a good approximation.

5. (a)

$$A = \begin{bmatrix} -18 & 13 & -4 & 4 \\ 2 & 19 & -4 & 12 \\ -14 & 11 & -12 & 8 \\ -2 & 21 & 4 & 8 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 528 & -392 & 224 & -176 \\ -392 & 1092 & -176 & 536 \\ 224 & -176 & 192 & -128 \\ -176 & 536 & -128 & 288 \end{bmatrix}$$

Eigenvalues of $A^T A$ are: 1600,400, 100 and 0

Eigenvectors corresponding to non-zero eigenvalues of $A^T A$ are:

$$v_1 = \begin{bmatrix} -2/5 \\ 4/5 \\ -1/5 \\ 2/5 \end{bmatrix} \quad v_2 = \begin{bmatrix} 4/5 \\ 2/5 \\ 2/5 \\ 0.2 \end{bmatrix} \quad v_3 = \begin{bmatrix} 2/5 \\ -1/5 \\ -4/5 \\ 2/5 \end{bmatrix} \text{ to which a fourth vector } v_4 = \begin{bmatrix} -1/5 \\ -2/5 \\ 2/5 \\ 4/5 \end{bmatrix}$$

can be added to form an orthonormal basis of R^4.

The first three columns of the U matrix are calculated from the V matrix columns and non-zero singular values using the $u_i = \sigma_i^{-1} A v_i$ formula. The last column of U is chosen so all the columns form an orthonormal basis for R^4. This gives the U matrix:

$$\begin{bmatrix} -0.5 & 0.5 & 0.5 & -0.5 \\ -0.5 & -0.5 & -0.5 & 0.5 \\ -0.5 & 0.5 & -0.5 & 0.5 \\ -0.5 & -0.5 & 0.5 & 0.5 \end{bmatrix}$$

S matrix would be:

$$\begin{bmatrix} 40 & 0 & 0 & 0 \\ 0 & 20 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and V matrix would be:

$$\begin{bmatrix} 0.4 & -0.8 & -0.4 & 0.2 \\ -0.8 & -0.4 & 0.2 & 0.4 \\ 0.2 & -0.4 & 0.8 & -0.4 \\ -0.4 & -0.2 & -0.4 & -0.8 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 6 & -8 & -4 & 5 & -4 \\ 2 & 7 & -5 & -6 & 4 \\ 0 & -1 & -8 & 2 & 2 \\ -1 & -2 & 4 & 4 & -8 \end{bmatrix}$$

Use Matlab to calculate the eigenvalues of $A^T A$ (to 4 decimals): 270.8673, 147.8538, 23.7266, 18.5522, 0 and corresponding eigenvectors:

$$v_1 = \begin{bmatrix} 0.1002 \\ -0.6064 \\ 0.2131 \\ 0.5217 \\ -0.5520 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.3892 \\ 0.2867 \\ 0.8419 \\ -0.1412 \\ -0.1940 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.7353 \\ 0.2682 \\ 0.07251 \\ -0.3772 \\ -0.4897 \end{bmatrix} \quad v_4 = \begin{bmatrix} -0.4057 \\ 0.4953 \\ -0.4518 \\ 0.2258 \\ -0.5787 \end{bmatrix}$$

and $v_5 = \begin{bmatrix} 0.3649 \\ 0.4825 \\ 0.1910 \\ 0.7174 \\ 0.2879 \end{bmatrix}$

The U matrix is obtained using the first 4 (non-zero) singular values, corresponding columns of V and the usual formula:

$$\begin{bmatrix} 0.5721 & -0.6518 & 0.4207 & -0.2661 \\ -0.6348 & -0.2393 & 0.6754 & 0.2891 \\ -0.07041 & -0.6326 & -0.5301 & 0.5602 \\ 0.5145 & 0.3430 & 0.2930 & 0.7292 \end{bmatrix}$$

S matrix would be:

$$\begin{bmatrix} 16.46 & 0 & 0 & 0 & 0 \\ 0 & 12.16 & 0 & 0 & 0 \\ 0 & 0 & 4.871 & 0 & 0 \\ 0 & 0 & 0 & 4.307 & 0 \end{bmatrix}$$

and V matrix would be:

$$\begin{bmatrix} 0.1002 & -0.3892 & 0.7353 & -0.4057 & 0.3649 \\ -0.6064 & 0.2867 & 0.2682 & 0.4953 & 0.4825 \\ 0.2131 & 0.8419 & 0.07251 & -0.4518 & 0.1910 \\ 0.5217 & -0.1412 & -0.3772 & 0.2258 & 0.7174 \\ -0.5520 & -0.1940 & -0.4897 & -0.5787 & 0.2879 \end{bmatrix}$$

6. (a)

$$A = \begin{bmatrix} 4 & 0 & -7 & -7 \\ -6 & 1 & 11 & 9 \\ 7 & -5 & 10 & 19 \\ -1 & 2 & 3 & -1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 102 & -43 & -27 & 52 \\ -43 & 30 & -33 & -88 \\ -27 & -33 & 279 & 335 \\ 52 & -88 & 335 & 492 \end{bmatrix}$$

Eigenvalues of A^TA are (4 decimals): 749.9785, 146.2009, 6.8206, 0.00000144.
The corresponding eigenvectors of A^TA are:

$$v_1 = \begin{bmatrix} -0.04893 \\ 0.1277 \\ -0.5782 \\ -0.8043 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0.8186 \\ -0.3348 \\ -0.4216 \\ 0.2001 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.5715 \\ 0.4497 \\ 0.5742 \\ -0.3761 \end{bmatrix} \quad \text{and } v_4 = \begin{bmatrix} 0.02846 \\ 0.8182 \\ -0.3978 \\ 0.4142 \end{bmatrix}$$

Using the usual relationship to get the columns of U from those of V and the non-zero singular values we have:

$$u_1 = \begin{bmatrix} 0.3462 \\ -0.4812 \\ -0.8050 \\ -0.02286 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0.3990 \\ -0.6685 \\ 0.5782 \\ -0.2442 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0.3446 \\ -0.01872 \\ 0.1330 \\ 0.9291 \end{bmatrix} \quad u_4 = \begin{bmatrix} -0.7760 \\ -0.5668 \\ -0.002831 \\ 0.2768 \end{bmatrix}$$

The expanded decomposition for A is

$$A = \sqrt{749.9785}u_1v_1^T + \sqrt{146.2009}u_2v_2^T + \sqrt{6.8206}u_3v_3^T + \sqrt{0.00000144}u_4v_4^T$$

Zeroing the smallest singular value, we get:

$$A' = 27.386u_1v_1^T + 12.091u_2v_2^T + 2.612u_3v_3^T = \begin{bmatrix} 4.0 & 0.0007342 & -7.0 & -7.0 \\ -6.0 & 1.001 & 11.0 & 9.0 \\ 7.0 & -5.0 & 10.0 & 19.0 \\ -1.0 & 2.0 & 3.0 & -1.0 \end{bmatrix}.$$

To evaluate the difference between A and A' use $||A - A'|| = 0.0012$.
The rank of A is 4 (but it's close to singular) and rank of A' is 3.

Now suppose $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and solve $Ax = b$ to get $x = \begin{bmatrix} -19 \\ -573 \\ 280 \\ -291 \end{bmatrix}$

The condition number of A is 23680. A matrix with a large condition number is ill-conditioned and is very sensitive to round off errors. The condition number of $A' = 6.175 \times 10^{16}$, i.e. A' is singular to machine precision. Therefore there is no unique solution to $A'x = b$. Accumulated roundoff error can provide a meaningless (large) solution (as in Matlab). The matrix A' provides the singular matrix to which A is close. Numerical algorithms have difficulty distinguishing between a matrix such as A, which is 'close-to-singular' and A' which is singular. Hence the behaviour of A is ill-conditioned.

(b) $$A = \begin{bmatrix} 5 & 3 & 1 & 7 & 9 \\ 6 & 4 & 2 & 8 & -8 \\ 7 & 5 & 3 & 10 & 9 \\ 9 & 6 & 4 & -9 & -5 \\ 8 & 5 & 2 & 11 & 4 \end{bmatrix}$$

Eigenvalues of A^TA are: 672.5891, 280.7447, 127.5031, 1.1632, 0.00000016 with corresponding eigenvectors:

$$v_1 = \begin{bmatrix} -0.4723 \\ -0.3094 \\ -0.1440 \\ -0.7115 \\ -0.3927 \end{bmatrix} \quad v_2 = \begin{bmatrix} -0.5883 \\ -0.3921 \\ -0.2463 \\ 0.2789 \\ 0.6015 \end{bmatrix} \quad v_3 = \begin{bmatrix} 0.2432 \\ 0.1632 \\ 0.1302 \\ -0.6428 \\ 0.6957 \end{bmatrix} \quad v_4 = \begin{bmatrix} 0.4707 \\ -0.2050 \\ -0.8567 \\ -0.04910 \\ -0.001418 \end{bmatrix}$$

and $$v_5 = \begin{bmatrix} 0.3875 \\ -0.8257 \\ 0.4094 \\ 0.01930 \\ -0.0005662 \end{bmatrix}$$

Corresponding columns of U (obtained from Matlab or otherwise) are:

$$u_1 = \begin{bmatrix} -0.4607 \\ -0.2664 \\ -0.6144 \\ 0.06491 \\ -0.5788 \end{bmatrix} \quad u_2 = \begin{bmatrix} 0.1791 \\ -0.4877 \\ 0.08266 \\ -0.8445 \\ -0.1006 \end{bmatrix} \quad u_3 = \begin{bmatrix} 0.3186 \\ -0.7382 \\ 0.2429 \\ 0.5310 \\ -0.1121 \end{bmatrix} \quad u_4 = \begin{bmatrix} 0.4868 \\ -0.08438 \\ -0.7459 \\ 0.02588 \\ 0.4460 \end{bmatrix}$$

and $$u_5 = \begin{bmatrix} -0.6458 \\ -0.3730 \\ 0.01883 \\ 0.0009253 \\ 0.6659 \end{bmatrix}$$

The expanded decomposition of A is:

$$A = 25.9343 u_1 v_1^T + 16.7554 u_2 v_2^T + 11.2917 u_3 v_3^T + 1.0785 u_4 v_4^T + 0.0004 u_5 v_5^T$$

Zeroing the smallest singular value gives:

$$A' = 25.9343 u_1 v_1^T + 16.7554 u_2 v_2^T + 11.2917 u_3 v_3^T + 1.0785 u_4 v_4^T = \begin{bmatrix} 5.0 & 3.0 & 1.0 & 7.0 & 9.0 \\ 6.0 & 4.0 & 2.0 & 8.0 & -8.0 \\ 7.0 & 5.0 & 3.0 & 10.0 & 9.0 \\ 9.0 & 6.0 & 4.0 & -9.0 & -5.0 \\ 8.0 & 5.0 & 2.0 & 11.0 & 4.0 \end{bmatrix}.$$

To evaluate the difference between A and A' use $||A - A'|| = 0.0003779$

The rank(A) = 5 and rank(A')=4 as expected due to the zeroing of a small non-zero singular value.

Now using $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$ and solving $Ax = b$ gives $x = \begin{bmatrix} 2049 \\ -4365 \\ 2164 \\ 102 \\ -3 \end{bmatrix}$

Using a different $b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5.02 \end{bmatrix}$ very close to the first choice, and solving $Ax = b$ again gives a very different solution $x = \begin{bmatrix} 2186 \\ -4656 \\ 2308 \\ 109 \\ -3 \end{bmatrix}$

A is close-to-singular and very sensitive to round off error.

The matrix A' is singular (to machine precision) therefore there is no unique solution to $A'x = b$. Accumulated roundoff error can provide a meaningless (large) solution (as in Matlab). The condition number of $A = 68620$ and condition number of $A' = 1.791 \times 10^{16}$ (i.e. A' is singular). The matrix A' provides the singular matrix to which A is close.